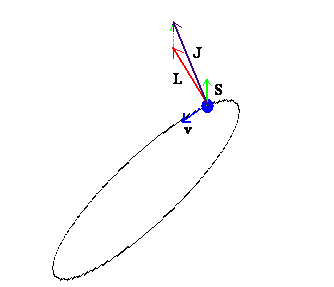
**Total Angular Momentum**

Now that we’ve discussed orbital angular momentum, **L**, and spin angular momentum, **S**, it seems appropriate to discuss a particle’s total angular momentum, **J** (running out of letters I guess).



Illustrated below:



Just like L and S are quantized, J is too of course. And we’d like to determine what possible values J can assume. But we’ve already answered this question actually. **J** satisfies the same commutation relations as **L** and **S** do, namely:



which follows from the **L** and **S** commutation relations. Since the different components of **J** do not commute with each other, it follows that we cannot know all three simultaneously, just as was true for **L** and **S**. So for instance if we know Jz, then we cannot know either Jx or Jy. But we can know J2 since [J2,Jz] = 0, just as was true for **L** and **S**. What are the simultaneous eigenvalues/eigenvectors of Jz and J2? Let’s call them |jmj>. Then it follows from our earlier algebraic analysis of the eigenvectors of Lz and L2, that:



For review, I’ll rewrite the details of the derivation. We start with creating the raising operator/lowering operators which increase/decrease the value of the z-component of the total angular momentum. In analogy with before, they are:



So we have:



What is the action of this operator on our eigenkets? As we might imagine, the same as the analogous operators for L and S – it raises/lowers the total angular momentum z-component by ћ. To verify, use the commutation relation:



(easy to prove from the commutation relations­) and repeat our arguments from before:



So ± raises/lowers the eigenvector to one with one more/less z-component angular momentum quanta. What does it do to the magnitude of the angular momentum? To determine that use the commutation relation (also easy to prove from the commutation relations)



and then we have:



So the raising/lowering operators do not change the total angular momentum. So all in all we have:



Let’s work out the proportionality constant now. So say the constant of proportionality is c so that:



Then take the dot product of this vector with itself and we can work it out. We have:



Now we need to find another expression for what the product of these two operators is in order to explicitly evaluate their action (including the constant of proportionality) on the kets. So consider that (can prove just like before):



Now use this in our normalization expression,



and so we have explicitly that:



It is clear from this that the maximum mj can be is simply j, and the minimum it can be is –j. And moreover, one must be able to raise the state from mj = - j to mj = j through a whole number of steps and so we must have that: 2j+1 = a natural number, N, which implies that j = 0, ½, 1, 3/2, 2, 5/2, etc. repeating our arguments from before.

**Uncoupled description of particle**

When describing the angular momentum of a particle, there are usually two prominent choices to make. The first is called the uncoupled representation of the particle. In this representation we seek knowledge of the particle’s L2, Lz, S2, and Sz values. We can know all of these things because each of these operators commutes with the other. And we cannot be more specific about the angular momentum of the particle than this because there are no other angular momentum operators that commutes with these four. The eigenvectors/eigenvalues of these operators are designated |ℓmℓsms> = |ℓmℓ>|sms> and they satisfy:



It is called the uncoupled representation because these eigenstates have information about L and Sseparately, while no knowledge of the J is given.

**Example**

Suppose we have an electron in the spin up state with angular momentum ћ√2, with z-component angular momentum of -ћ . What is its eigenstate? What is its wavefunction?

Since L = ћ√2, then L2 = 2ћ2 → ℓ = 1. And mℓ = -1 if Lz = -ћ. An electron has s = ½ , and ms = ½ (because its spin-up). So its eigenstate is:



Projecting this onto the position/spin basis we have:



**Example**

What is the value of the z-component of the total angular momentum of this particle?

So I guess we actually can know a *little* bit more about the state of the particle. Operating Jz on it we have,



So if we measured the z-component of its total angular momentum we would come out with –ћ/2. Can we determine J2? Let’s try:



But this is as far as we can go. Since we don’t (indeed cannot) know Lx, Ly, Sx, and Sy we cannot obtain J2.

**Coupled representation**

In contrast there is the so-called coupled representation of a particle’s angular momentum. Here we look for eigenstates of the following operators L2, S2, J2 and Jz. As before, this is the maximal amount of (different) information that we can extract from the particle’s state. This is called the coupled representation because it gives information about J, which ‘couples’ L and S. We would denote these eigenstates by: |ℓsjmj>, and they should satisfy,



Before we go on, though, we should be sure that we can have such states. We should be sure that all the operators do commute. Let’s check. Well L2 and S2 commute with each other because they live in different Hilbert Spaces. L2 lives in the position space |**r**>, while S2 lives in the spin space |sms> so these operators do not interfere with each other. What about L2 and J2?



So this works. Let’s check,



Similarly S2 commutes with J2 and Jz. And finally,



So everything checks out. Now, just like specifying ℓ and s restricts what mℓ and ms can be, the same happens to j and mj. So we’d like to know what possible values j and m­j may assume for these eigenvectors. To that end, we’ll attempt to solve the equations which should tell us.

There is a general argument we can make. Let’s go back to the uncoupled representation. We know that there are a total of (2ℓ+1)(2s+1) basis states, for a given ℓ and s. And so the coupled representation must have the same number of basis states. Further, we know that mj runs between -j and +j, so that there are 2j+1 possible states associated with a given j. We may imagine that the largest j can be is when the angular momentum and spin are completely aligned, so jmax = ℓ + s, and the minimum j would be is when they are completely unaligned so that jmin = ℓ - s. Well, actually jmin must be a whole number (non-negative), so let’s say jmin = |ℓ - s|. And we can prove that this prescription on the allowed values of j and mj does match this number of states. For the total number of states would be (assume ℓ > s for the sake of discussion):



So there you go. And we can write:



But we need now to figure out what the coefficients cmℓms are.

**Example**

A spin-1/2 particle has the angular wavefunction:



where Yℓm(θ,φ) are the normalized spherical harmonics. What are the possible results of measuring the particle’s total spin quantum numbers j and mj?

Guess I’d break each part of the wavefunction into its possible |jmj> states. The first guy has ℓ = 3, s = ½, and so possible j-values of 5/2, 7/2. And so possible mj values ranging from -5/2 to 5/2 or from -7/2 to 7/2. But we’ll establish later that we must have mj = mℓ + ms. So in this state, really, mj must be 0 ± ½ = ±½ (because we don’t know if the spin is up or down). The second guy has possible j-values of 3/2, 5/2 and mj values would ostensibly run between -3/2 and 3/2 or from -5/2 to 5/2. But again, we’ll establish later that we must have mj = mℓ + ms. So we must have, in this state, mj = mℓ + ms = 1 ± (1/2) = ½, 3/2.